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Heisenberg Equation of Motion Approach to Non-Hermitian Hamiltonians with Real Spectrum

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Abstract

In this paper, three examples of non-Hermitian Hamiltonians were presented on which an approach was applied based on the Heisenberg equation of motion, namely a first-order equation in the coordinate and momentum.

Subject Areas

Mathematics

Keywords

Non-Hermitian Hamiltonians, The Coordinate and Momentum

1. Introduction

For some time, non-Hermitian Hamiltonians have been at the center of intense scientific activities of research in theoretical physics. Our motivation to write this paper comes from a recent work in which the non-Hermitian Hamiltonians are treated by an approach that is presented in Ref. [1]. We know that in quantum mechanics, observables are realized in terms of self-adjoint operators on the Hilbert space. It is for these operators that the spectral theorem holds in Ref. [2]. However, in 1998 in Ref. [3], it was shown that a non-Hermitian Hamiltonians can still have an entirely real spectrum provided that it possesses PT symmetry. In several articles, one also finds other approaches aimed at solving non-Hermitian models. These approaches are, for instance, perturbation theory, with a variety of models in Refs. [4]-[8].

In this paper, we treat three examples of non-Hermitian Hamiltonians drawn from the cited references. The main idea is to apply an old approach, namely the Heisenberg equation of motion, and then to determine the real spectrum for these examples. In our analysis, we also remarked that only the second example possesses an energy spectrum multiplied through a complex phase. We construct a general form of the Hamiltonian that generates the three examples treated in this paper. We should note that by introducing a local similarity transformation, it is possible to put together the spectrum of either two related Hamiltonians. This Hamiltonian general form can be diagonalized through an algebraic approach namely algebra bi-Fock, and this algebra is constructed on the Hilbert space.

We investigate three simple examples of the non-Hermitian Hamiltonians drawn in the literature in order to solve the real energy spectrum. In Sec.2, we solve the real spectra of the non-Hermitian systems and one remarks that for the second example, one obtains another result namely an energy spectrum multiplied through a complex phase. We give a much more general Hamiltonian formulation in which one finds the three non-Hermitian systems. And we have introduced that, through a local similarity transformation, it is possible to establish a link between the spectrums of either two related Hamiltonian. We summarize our results in Sec.4.

2. Heisenberg Equation of Motion with Real Spectrum

Here, we consider three examples of non-Hermitian Hamiltonians drawn in Refs. [7] [8] that we develop an approach based on the Heisenberg equation of motion, in order to determine the real spectrum. One shows that among these three examples chosen only one possesses a spectrum that is multiplied through a complex phase. We have introduced a general form of the Hamiltonian that generates all the three examples proposed in this work.

2.1. Extended Harmonic Oscillator

In this subsection, we recall the form of this extended harmonic oscillator given in Ref. [6]

$$H_\beta = \frac{\beta}{2}\left(p^2 + x^2\right) + i\sqrt{2p}, \quad \beta > 0 \tag{1}$$

where H_β is non-Hermitian, and x and p the canonical coordinate and momentum, which obey the following canonical commutation relations

$$[x, p] = i\hbar, \quad [x, x] = 0, \quad [p, p] = 0. \tag{2}$$

Within these canonical commutation relations, we have suppressed $\hbar$ in order to avoid all factors related to $\hbar$ in our results. Let us recall that this Hamiltonian is not Hermitian, *i.e.* $H_\beta^\dagger \neq H_\beta$, and it is also not PT symmetric,

$$H_\beta = H_\beta^{PT} : (PT)^{-1} H_\beta (PT). \tag{3}$$

In Equation (3) we have used the conventional definitions from Ref. [1], with P

and T representing the parity and time-reserval transformations given by

$$\begin{aligned} P &: x \to -x, \quad p \to -p, \quad i \to +i \\ T &: x \to +x, \quad p \to -p, \quad i \to -i. \end{aligned} \tag{4}$$

By considering the Equation (1), we will find the solution by deriving the position operator x. The time evolution of the system is described by

$$\dot{x} = i\left[H_\beta, x\right], \;\; \dot{p} = i\left[H_\beta, p\right]. \tag{5}$$

this Equation (5) is known as the Heisenberg equation of motion for position and momentum operators. From Equations (1) and (2), we get

$$\dot{x} = \beta p - \sqrt{2i}, \quad \dot{p} = -\beta x. \tag{6}$$

The main idea is to eliminate the momentum operator in Equation (6) in order to obtain the second order equation of motion in terms the position operator x,

$$\ddot{x} + \beta^2 x = 0. \tag{7}$$

It is clear that this is a second order equation of motion that describes the harmonic oscillator of frequency β. From this Equation (7), we can now write the Hermitian Hamiltonian under the form

$$h_\beta = \frac{1}{2}P^2 + \frac{1}{2}\beta^2 x^2, \tag{8}$$

this Equation (8) corresponds to the following spectrum

$$E_\beta = \beta\left(n + \frac{1}{2}\right). \tag{9}$$

2.2. Swanson Hamiltonian

In this section we consider a Hamiltonian given in Refs. [9]-[12]. The goal is to solve the Swanson Hamiltonian with the Heisenberg equation of motion approach in order to determine its energy spectrum. The Hamiltonian form is given by

$$H_0 = \frac{1}{2}\left(p^2 + x^2\right) - \frac{i}{2}\left(\tan 2\theta\right)\left(p^2 - x^2\right), \tag{10}$$

where θ is a real parameter, with $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. The position and conjugate momentum operators satisfy the commutation rules in Equation (2).

It is clear that this Hamiltonian is not Hermitian, $H_\theta^\dagger \neq H_\theta$, or

$$H_\theta = H_\theta^{PT} := \left(PT\right)^{-1} H_\theta \left(PT\right), \tag{11}$$

where P and T designate the parity and time-reversal transformations as defined in Equation (4), respectively. Then, the Heisenberg equation of motion for position and momentum is of the same form as in Equation (5).

By taking the Equations (5) and (2), one obtains

$$\dot{x} = \left(1 - i\left(\tan 2\theta\right)\right)p, \quad \dot{p} = \left(1 + i\left(\tan 2\theta\right)\right)x. \tag{12}$$

We introduce a complex factor α under the form

$$\alpha = 1 - i\left(\tan 2\theta\right) = \frac{1}{\cos 2\theta}\mathrm{e}^{-2i\theta} \tag{13}$$

Finally, to obtain the equation of the motion, we should eliminate the momentum operator p in Equation (12), which leads to an equation of motion in terms of the position x

$$\ddot{x}+\alpha^2 x=0. \tag{14}$$

This equation of the motion corresponding to an Hermitian Hamiltonian of the form

$$h_\theta=\frac{1}{2}p^2+\frac{1}{2}\alpha^2 x^2, \tag{15}$$

whose energy spectrum corresponds to the Swanson Halmitonian given in Refs. [6] [7]. This Hamiltonian h_θ must have the form

$$E_\theta=\frac{1}{\cos 2\theta}\mathrm{e}^{-2\theta}\left(n+\frac{1}{2}\right),\quad \omega_\theta=\frac{1}{\cos 2\theta}\mathrm{e}^{-2\theta}, \tag{16}$$

and this equation becomes,

$$E_\theta=\omega_\theta\left(n+\frac{1}{2}\right). \tag{17}$$

2.3. Swanson Model

Here, we shall take the Swanson model in its standard form, which is given in Ref. [8],

$$H=\frac{p^2}{2m_1}+\frac{1}{2}i\omega\epsilon\{x_r,p_r\}+\frac{1}{2}m_1\omega^2x^2, \tag{18}$$

with $m_2=\left(1-\epsilon^2\right)m_1$.

For applying our approach, which consists of first finding the Heisenberg equation of motion.

We will also consider that this amounts to completing the square as

$$H=\frac{\left(p+i\epsilon m_1\omega x\right)^2}{2m_1}+\frac{1}{2}m_1\omega^2x^2, \tag{19}$$

where m and ω are the mass and angular frequency of the harmonic oscillator. (x,p) is a pair of canonical coordinate and momentum, and satisfies the commutation rule given in Equation (2). The Swanson model consists of two contributions: a real and an imaginary term, so we have $H^\dagger=H$ and the Hamiltonian is not Hermitian, but it is PT symmetric,

$$H=H^{PT}:=\left(PT\right)^{-1}H\left(PT\right), \tag{20}$$

where P and T designate the transformation given in Equation (4). By using the equation of motion in the Equations (5) and (2), we have

$$\ddot{x}+\left(1+\epsilon^2\left(\omega^2-1\right)\right)x=0. \tag{21}$$

It is easy to write the Hermetian Hamiltonian associated with this equation of motion, whose form is

$$h=\frac{p^2}{2m_1}+\frac{1}{2}m_1\left(1+\epsilon^2\left(\omega^2-1\right)\right)x^2, \tag{22}$$

This Hermitian Hamiltonian in Equation (22) has an energy spectrum

$$E_n = \sqrt{1+\epsilon^2\left(\omega^2-1\right)}\left(n+\frac{1}{2}\right), \quad n=0,1,2,\cdots \tag{23}$$

This energy spectrum corresponds to the Swanson model, with an important multiplicative factor.

3. General Hamitonian

In this section, we give the most general form of the Hamiltonian for the three simple examples discussed in the previous sections, which allows us to put them together. Before, we introduced the usual x and p operators that define the Heisenberg algebra. These operators are hermitians, $x^\dagger = x$ and $p^\dagger = p$. The explicit form of the general Hamiltonian is

$$H = \frac{1}{2\alpha}p^2 + \alpha V(x) + \frac{1}{2}i\left(pW(x) + W(x)p\right), \tag{24}$$

where $V(x)$ and $W(x)$ two real functions expressed in terms of the variable x, and with the complex factor α from Equation (13).

Let us rewrite the Equation (24) under the form

$$H = \frac{1}{2\alpha}\left(p + i\alpha W(x)\right)^2 + \alpha\left(V(x) + \frac{1}{2}W(x)^2\right) = \frac{1}{2\alpha}P^2 + \left(V(x) + \frac{1}{2}W(x)^2\right)^2, \tag{25}$$

with the definition

$$X = x, \quad P = p + i\alpha W(x). \tag{26}$$

Let us note that $X^\dagger = X$ remains hermitian (self-adjoint), but that $P^\dagger \neq P$ is not hermitian if $W(x) \neq 0$, while these two operators define again a Heisenberg algebra,

$$\left[X, X\right] = 0, \quad \left[X, P\right] = i\hbar, \quad \left[P, P\right] = 0 \tag{27}$$

This directly implies that the Heisenberg equations obtained for $X = x$ from either of these forms of H, once p or P are eliminated, are similars. This can be explicitly verified. It is an established result in Ref. [1], obtained difficult in the case of a series representation from Taylor of $W(x)$. But it is an immediate result and valid for all differentiable function $W(x)$. The second-order equation of motion is simply,

$$\ddot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(V(x) + \frac{1}{2}W(x)^2\right) \tag{28}$$

under this form independent of α. By substituting Equation (26) into Equation (25), we get

$$H = \frac{1}{\alpha}\left(\frac{1}{2}p^2 + \alpha^2\left(V(x) + \frac{1}{2}W(x)^2\right)\right) = \frac{1}{\alpha}H_0 \tag{29}$$

it is clear that H and H_0 lead to the same second-order equation of motion in x. However their energy spectra are not similar, because they differ through the normalisation factor $\frac{1}{\alpha}$. It is not incorrect to say or think that if the equations of motion are similar, then the spectrums are also similar, at a near additive constant.

Nevertheless, the real character of the spectrum of H can be established as follows, through a argument that is not related to the Heisenberg equation of motion for x. In complement at the result of the Ref. [1], let us consider now the situation concerning spectrum the real character of H. Let us introduce partner Hamiltonian to H, expressed in term of x and p, and not X and P,

$$H_1 = \frac{1}{2\alpha} p^2 + \alpha \left(V(x) + \frac{1}{2} W(x)^2 \right) \tag{30}$$

We shall establish that if the spectrum of H_1 is real, then the spectrum of H is also real, even if H is not hermitian. In particular, if α is real and corresponds to a mass factor, $\alpha = m > 0$, then clearly the spectrum of H_1 is real. This result is valid for all choices of the real functions $V(x)$ and $W(x)$. Let us consider a function $\Phi(x)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x} \Phi(x) = \alpha W(x) \tag{31}$$

where $\Phi(x)$ is primitive of $\alpha W(x)$. Now consider a quantum state $|\psi_\lambda\rangle$ eigenstate of H, with possibly complex eigenvalue E_λ,

$$H|\psi_\lambda\rangle = E_\lambda |\psi_\lambda\rangle \tag{32}$$

Let us then introduce the quantum state

$$|\varphi_\lambda\rangle = \mathrm{e}^{-\frac{1}{\hbar}\Phi(x)} |\psi_\lambda\rangle, \quad |\psi_\lambda\rangle = \mathrm{e}^{-\frac{1}{\hbar}\Phi(x)} |\varphi_\lambda\rangle \tag{33}$$

In calculating then the action from H over $|\psi_\lambda\rangle$ expressed in term of $|\varphi_\lambda\rangle$, it is immediate to establish that the state $|\varphi_\lambda\rangle$ is an eigenstate of H_1 with eigenvalue E_λ,

$$H_1 |\varphi_\lambda\rangle = E_\lambda |\varphi_\lambda\rangle \tag{34}$$

However if the spectrum of the eigenvalue of H_1 is real then indeed the eigenvalues spectrum of H is also real, while H is not hermitian, and this without H being PT-symmetric. This completes the discussion of the Ref. [1] in a general way on this result in particular.

However, a technical questions arising from and not considered in Ref. [1] concerns the definition area of the operators. One works on a Hilbert space with normalisable states $|\psi_\lambda\rangle$. This also touches on the boundary conditions to impose at infinity to the corresponding wave functions.

In the case of a non-Hermitian Hamiltonian this can have significant consequences. As well as if the spectrum of H_1 is real, of the function $W(x)$, it is not certain that the states $|\psi_\lambda\rangle$ are again normalisable. It would be interesting to study this question in the case of a simple example with linear equations, where $V(x)$ is quadratic in x and $W(x)$ contains a constant and linear term in x.

Even if Ref. [1] is not correct to say that similar equations of motion mean similar spectra, it is true that the discussion above which generalises and completes theirs, shows that it is possible to have real spectra for non-Hermitian Hamiltonians which are PT-symmetric. The result emphasize a kind of transformation (33) that is similar to a local phase transformation of a scale transformation, or similarity transformation in general, which yields the general property of the non-Hermitian Hamiltonians but real spectrum.

Indeed in the case where $V(x)$ is quadratic in x (harmonic oscillator) and $W(x)$ is linear in x (constant and linear in x and α complex), it is possible to diagonalize H_1 by means of bi-Fock algebras in Refs. [9] [10], namely operators which are not adjoint, e.g. annihilation and creation operators, but that nevertheless obey the same kind of Fock algebra. Then it is immediate to find the spectrum, even non-real, of H_1, and from this the correspondign spectrum of H, through the discussed transformations above.

Then the presence of the term in $W(x)$ in the relation that connects P and p, where we have $P = p + i\alpha W(x)$ translating definitions of these annihilation and creation operators, by a shift in the term in p—a shift that is linear in x. Since all rest is again linear, it is possible to solve for the energy spectrum and then for the corresponding wave function. However, it is not certain that these wave functions for the eigenstates of H will be normalisable.

Nevertheless, each of these eigenvalues of the Schr¨odinger equation is of the second order in x and possesses two independent linear solutions, of which one is obtained by the construction in term of the annihilation and creation operators of the bi-Fock type mentioned above. It remains then to construct the other linearly independent solution. It is still possible that one certain linear combination of the two solution is normalisable for a non-hermitian H, what should then completely solve the problem. It would be interesting to study the situation in the case of $\alpha = m > 0$ with $V(x)$ harmonic and $W(x)$ linear, in the spirit of Ref. [1] and the generalized results obtained above.

4. Conclusions

We have presented three simple examples of non-Hermitian systems in the framework of quantum mechanics and remarked that these Hamiltonians all have a real energy spectrum. This shows that this approach is easy to use and gives good results. In our analysis, we have remarked that among these three examples in particular only the second example possesses an energy spectrum that is multiplied through a complex phase. This proves that it is not always obvious for two different systems given by two different Hamiltonians to have the same equation of motion or the same energy spectrum.

Furthermore, we have given a general form of the Hamiltonian that generates these examples. We have also introduced a local similarity transformation that allows us to establish a link between the spectra of either two related Hamiltonians. This general form of the Hamiltonian can be solved through an algebraic approach, namely bi-Fock algebras. These bi-Fock algebras allow us to diagonalize the general form of the Hamiltonian and determine their physical spectra, which are real.

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Conflicts of Interest

The authors declare no conflicts of interest.

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